L'Hopital + Indeterminate Forms [redacted]

I. $\lim = \frac{0}{0}$ or $\frac{\infty}{\infty}$; use L'Hopital

II. $\lim = 0 \cdot \infty$ or $\infty - \infty$; rewrite as I.
* often this can be done with a common denominator

III. $\lim = \infty^0$, 1^∞, [redacted] Let $\lim = y$ and take the ln of both sides; then simplify.

Improper Integrals [redacted]

I. Infinite limits

II. Infinite discontinuity

- If $f(x)$ is continuous on $[a, \infty)$ then $\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$
- If $f(x)$ is continuous on $(-\infty, \infty)$, then $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^\infty f(x)\,dx$ where c is a real number.
- If the limit exists, the integral CONVERGES at the limit; if the limit doesn't exist, the integral DIVERGES.

The limit of a sequence $\{a_n\}$ is L, $\lim_{n\to\infty} a_n = L$. If such a limit exists, the sequence CONVERGES at L; if the limit doesn't exist, the sequence DIVERGES.

* Squeeze theorem:
  If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$
  and there exists an integer N such that $a_n \leq c_n \leq b_n$ for all $n > N$, then
  $\lim_{n\to\infty} c_n = L$

* If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

If the terms of a sequence are all nonincreasing or nondecreasing, it is MONOTONIC.

If ~~then~~ a sequence is bounded ABOVE ($M \geq a_n$) and BELOW ($N \leq a_n$), then it is BOUNDED.

* Sequences that are bounded and monotonic converge.

$\sum_{n=1}^{\infty} a_n$ ← N^{th} PARTIAL SUM

If this converges, the series converges. If $\{S_n\}$ diverges, the series diverges.

Geometric

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1$$

If $|r| > 1$, the series diverges.

N^{th} term convergence

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

If $\lim_{n\to\infty} a_n \neq 0$, $\sum_{n=1}^{\infty} a_n$ diverges. If $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n$ may or may not converge.

Integral

If $\int_a^{\infty} f(x)$, where $a_n = f(x)$

* diverges — so does $\sum a_n$
* converges — so does $\sum^{\infty} a_n$

P-series $\frac{1}{n^p}$

($\frac{1}{n}$ = harmonic series)

* if $p > 1$, $\sum^{\infty} a_n$ converges
* if $p \leq 1$, $\sum^{\infty} a_n$ diverges

Teloscopic

write out first few terms to determine the n^{th} partial sum, then find $\lim_{n\to\infty} S_n$.

Limit comparison

If $\lim_{n \to a} \frac{a_n}{b_n}$ = a finite number, than both either converge or diverge.

Direct Comparison

Compare a_n to b_n, where $0 \leq a_n \leq b_n$

* If $\sum^{\infty} b_n$ converges, than $\sum a_n$ converges
* If $\sum^{\infty} a_n$ diverges, than $\sum b_n$ diverges.

Alternating Series

Let $a_n > 0$. The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ converges if:

* $\lim_{n \to \infty} a_n = 0$
* $a_{n+1} \leq a_n$ (for a sufficent n)

Ratio test

Let $a_n > 0$.

$\sum^{\infty} a_n$ converges absoloutely if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ and diverges if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$. Inconclusive if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$.

Root

let $a_n > 0$.

$\sum^{\infty} a_n$ converges absolutely if $\sqrt[n]{|a_n|} < 1$

$\sum^{\infty} a_n$ diverges if $\sqrt[n]{|a_n|} > 1$

Inconclusive if $\sqrt[n]{|a_n|} = 1$

Absolute Convergence – if both the alternating and non-alternating forms converge. If not, the series is conditionally convergant

---

Taylor + Maclaurian Polynomials

If f has n derivitaves at c, the polynomial

$$P_n(x) = f(c) + \frac{f'(c)(x-c)^1}{1!} + \frac{f''(c)(x-c)^2}{2!} + \cdots + \frac{f^n(c)(x-c)^n}{n!}$$

is called the n^{th} taylor polynomial for f(c), centered at c. If c=0, it's a Maclurian polynomial.

Remainder

For a Taylor polynomial, $f(x) = P_n(x) + R(x)$

exact — approximate — remainder

The remainder (error) is:

$$|R_n(x)| = |f(x) - P_n(x)|$$

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

use the maximum value between x and c for $f^{n+1}(z)$

For an alternating series,

$|S - S_N| = |R_N| \leq a_{n+1}$. The error is the next term.

Power Series

A series of the form $\sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x^1 + a_2 x^2 + \dots + a_n x^n$ is a power series; $\sum_{n=0}^{\infty} a_n (x-c)^n$ is centered at c.

A power series can be viewed as a function with a domain for which it converges.

- c is always in the domain
- The domain is of a certian radius of convergence, R, from c.
  - → if the series only converges at c, r = 0
  - → if the series always converges, r = ∞
  - ~~the endpoints must be~~

To find the radius of convergence, evaluate $\lim_{x \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$

$* = 0,\ R = \infty$

$* = \infty,\ R = 0$

$* = X,\ R = 1$

The series will converge if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, so simplify that to find the radius of convergence.

→ The endpoints must be checked individually when determining interval of convergance.

Power series for important functions:

$$e^x = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \frac{X^4}{4!} + \cdots + \frac{X^n}{n!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n}{(2n!)} x^{2n} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n (x^{2n-1})}{(2n+1)!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

Circle

All points equidistant from a fixed point.

$$(x-h)^2+(y-k)^2=r^2$$

- center (h,k)
- radius $=r$

Parabola

All points that are equidistant from a fixed line (directrix) and a fixed point (focus) not on the line.

 or 

$$4p(y-k)=(x-h)^2$$

or

$$y-k=\frac{(x-h)^2}{4p}$$

directrix $y=k-p$

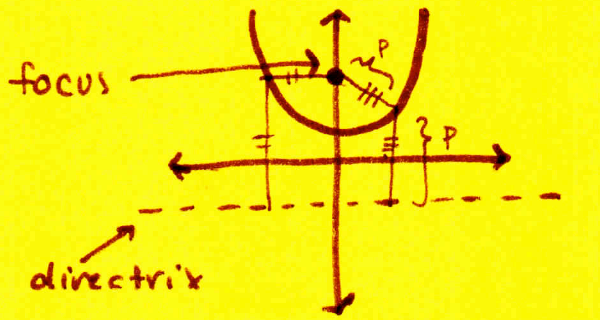
 or 

$$4p(x-h)=(y-k)^2$$

or

$$(x-h)=\frac{(y-k)^2}{4p}$$

directrix $= h-p$



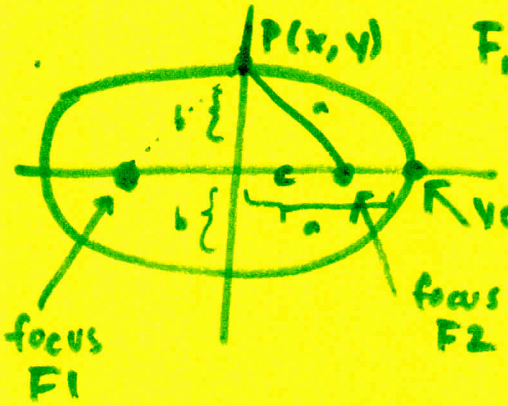
(distance = $\perp$ distance)

The midpoint between the focus and the directrix is the VERTEX.

The line that passes through the focus and the vertex is the AXIS OF SYMMETRY.

Ellipse

The set of all points (x, y) that the sum of the distances from two distinct points (foci) is constant.



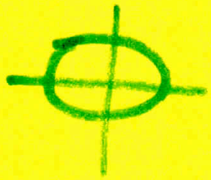
$F_1 \to P + F_2 \to P =$ a constant $= 2a$

$a^2 - b^2 = c^2$

vertex — the vertices lie on the major axis, a distance of a from the center. The perpendicular chord is the minor axis, with a length of 2b.

c units from center on major axis

* area: πab
* center (h, k)



$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

Ovalness = eccentricity $= e = \frac{c}{a}$

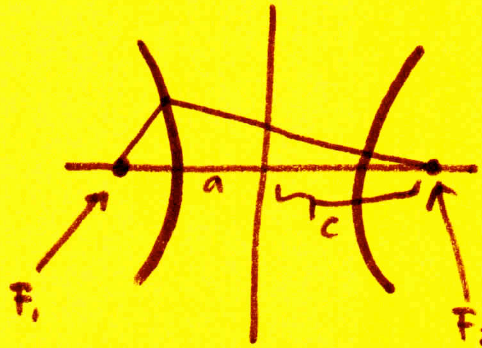
The eccentricity of an ellipse is always less than 1. If it is close to 0, it is circular; if is close to 1, it is more elongated.

בס״ד

Hyperbola

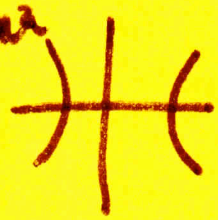
The set of all points (x, y) that the absolute value of the difference of the distances from two foci is constant.

* vertex: runs through foci and intersects hyperbola - these points
* line containing vertices = transverse axis
* other axis = conjugate axis
* midpoint of transverse axis = center, a distance of c from foci and a from vertices. $a^2+b^2=c^2$



$$|F_1 \to P - F_2 \to P| = 2a$$

* eccentricity: $e = \frac{c}{a}$, always larger than 1.



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

asymptotes: $y = k \pm \frac{b}{a}(x-h)$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

asymptotes: $y = k \pm \frac{a}{b}(x-h)$

Arc leangth:

$$s = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

for a parametric curve:

$$s = \int_{ta}^{tb} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Parametric Equations

are in terms of a third variable (the parameter).

The graph of these equations is called the plane curve.

If the equations are solved so that the parameter is eliminated, it is rectangular.

Motion ...

	Along a Line	Along a Curve
POSITION	$x(t)$	$\langle x(t), y(t)\rangle$
VELOCITY	$x'(t)$	$\langle x'(t), y'(t)\rangle$
ACCELERATION	$x''(t)$	$\langle x''(t), y''(t)\rangle$
SPEED	$\lvert x'(t)\rvert = \lvert v(t)\rvert$	$\sqrt{(x'(t))^2+(y'(t))^2}$
TOTAL DISTANCE	$\int_a^b \lvert v(t)\rvert\,dt$	$\int_a^b \sqrt{(x'(t))^2+(y'(t))^2}\,dt$

horozontial tangent line — slope = 0

vertical tangent line — undefined slope

If $f(t)=x$ and $g(t)=y$.

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dx}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$\frac{d^ny}{dx^n}=\frac{\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)}{\frac{dx}{dt}}$$